Indian Statistical Institute, Bangalore

M. Math. Second Year

Second Semester - Graph Theory and Combinatorics

Mid-Semester Exam Duration : 3 hours Max Marks 100 Date : Feb 20, 2017

Note: There are six questions, each carrying twenty marks. The maximum you can score is 100.

1. (a) State and prove the inclusion - exclusion formula.

   (b) For $k \geq 1$, let B_k be the total number of partitions of a k-set into non-empty cells. Put $B_0 = 1$. Then show that

$$B_k = \sum_{l=0}^{k-1} \binom{k-1}{l} B_l = \frac{1}{e} \sum_{n=0}^{\infty} \frac{n^k}{n!}$$

2. Let $N(n)$ denote the maximum possible number of mutually orthogonal partitions of a set of size n^2. Then show that

   (a) $N(n) \leq n+1$, with equality iff there is a projective plane of order n.

   (b) $N(n) \neq n$ for $n \geq 2$.

3. If $\mathbb{F}$ is a finite field of order q and n divides $q-1$ then show that

   (a) $\mathbb{F}^*$ has exactly $\frac{q-1}{n}$ nth powers.

   (b) For $a, b, c \in \mathbb{F}$, the equation $ax^2 + by^2 + cz^2 = 0$ has a non-trivial solution in $\mathbb{F}$.

4. (a) Show that every conic in $PG(2, q)$ is an oval.

   (b) Count the total number of ovals in a projective plane of order 4.

5. For a prime power q and $0 \leq k \leq n$, define $n!_q = \prod_{k=1}^{n} (q^k - 1)$ and $\binom{n}{k}_q = \frac{n!_q}{k!_q (n-k)!_q}$. Then show that $\binom{n}{k}_q$ is the total number of k-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$.
(Hint: First find the total number of $k-$tuples of linearly indep vectors in V.)

6. For a prime p, let K_p be the algebraic closure of the field Z/pZ.

   (a) Show that, for each $n \geq 1, K_p$ has a unique subfield L_n of order p^n.

   (b) Show that a subset L of K_p is a subfield if and only if $L = \cup\{L_n : n \in A\}$ for some set A of positive integers such that A is closed under taking *gcd* and *lcm*.